Bootstrapping Time Series Data

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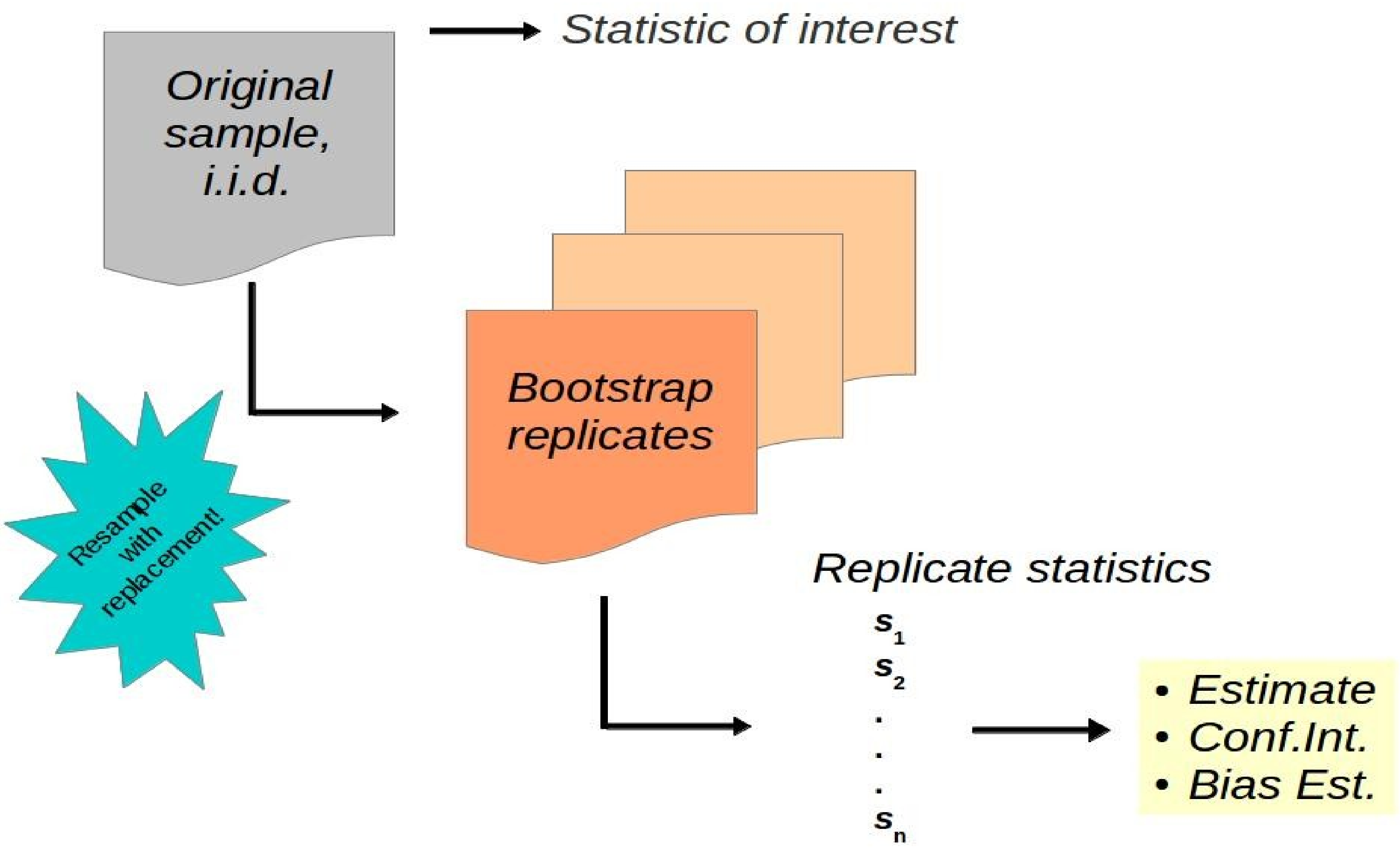
We'll cover a range of bootstrapping procedures today.

- Background on the bootstrap
- Non-parametric: The naïve bootstrap
- Handling dependency: The Moving Block bootstrap
- Honoring a model: Parametric bootstrap
- Balanced approach: The Maximum Entropy bootstrap

When and why do we bootstrap time series data?

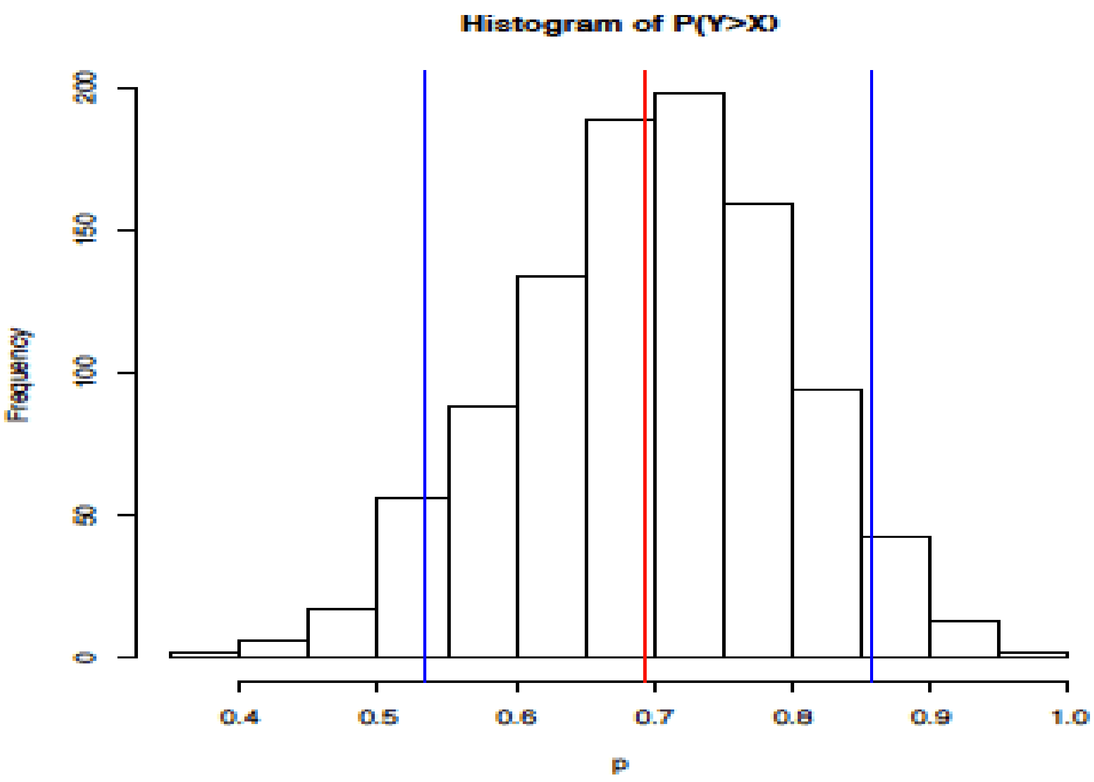
- You have some time series data
- But not much data – *whatever “much” means*
- Want to estimate a statistic – *especially a tricky statistic*
- . . . and its confidence interval
- No closed-form solution

Bootstrapping generates *bootstrap replicates* and *replicate statistics*.



Q: How do we get statistic's *conf. interval* from *replicate statistics*?

A: The percentiles of the empirical distribution (histogram) give the confidence interval for the statistic. Cool!



Bootstrapping *time series* data has special challenges.

- Interesting time series are not i.i.d.

  We difference the data.

- How do we generate plausible bootstrap replicates?

  Several ways. That's what this talk is really about.

- How do we deal with dependency structure?

  By choosing the right replication method. Stay tuned.

The bootstrap procedure requires i.i.d. data.

- i.i.d. necessary for resampling with replacement.
- Differencing time series can create i.i.d. data.
- Random walk model, where $\varepsilon_t \sim N(0, \sigma^2)$:

$$y_t = y_{t-1} + \varepsilon_t$$

- Becomes:

$$\varepsilon_t = y_t - y_{t-1}$$

If differences are i.i.d., we can use the *naïve bootstrap.*

Procedure:

1) Calculate successive *differences*.
2) Repeatedly,
    1) Resample the differences <u>with</u> replacement.
    2) Sum those differences to construct one replicate time series.
    3) Using that time series, calculate one replicate statistic.
3) From all the replicate statistics, form the estimate and confidence interval:

    Mean of replicate statistics → estimate

    Percentiles of replicate statistics → confidence interval

Toy Example

Given time series:

[1] 10.00 9.67 9.50 8.66 8.33 7.26 7.48 8.03 8.60 8.44

Statistic of interest for given data:

[1] 2.74

Compute differences:

[1] -0.33 -0.17 -0.84 -0.33 -1.07 0.22 0.55 0.57 -0.16

Resample the differences with replacement:

[1] 0.55 -0.16 -0.84 -0.33 0.22 -0.84 0.22 0.22 0.57

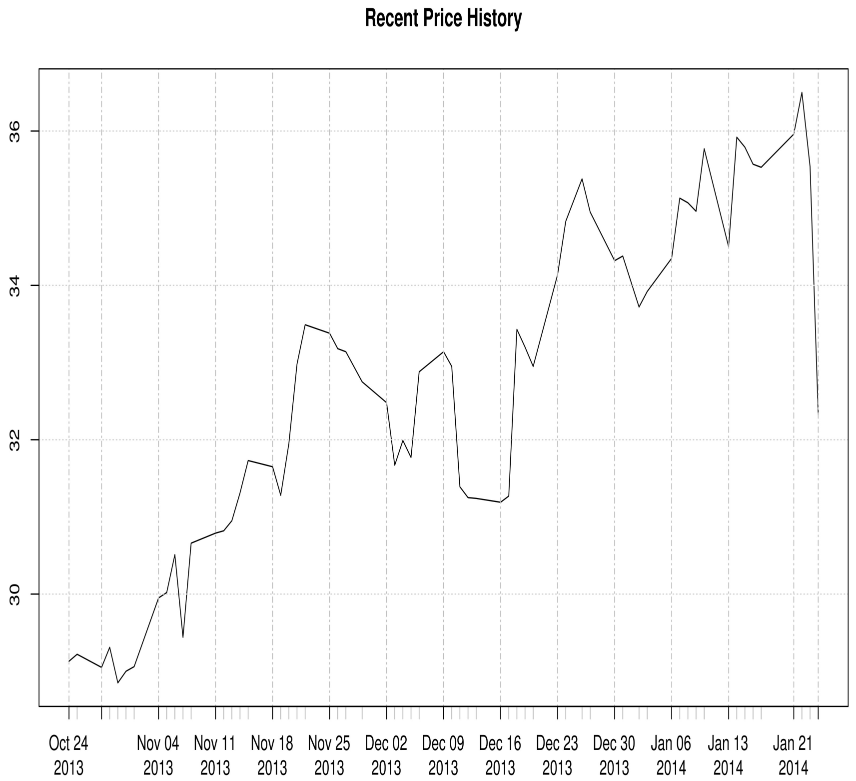
Construct one bootstrap replicate (by summing):

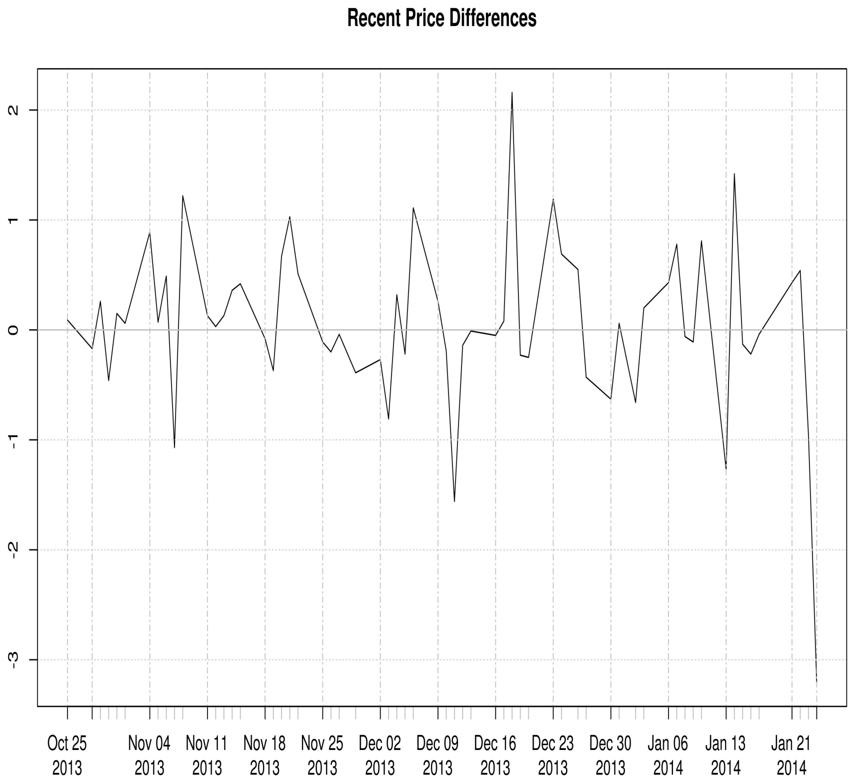
[1] 10.00 10.55 10.39 9.55 9.22 9.44 8.60 8.82 9.04 9.61

Compute one replicate statistic:

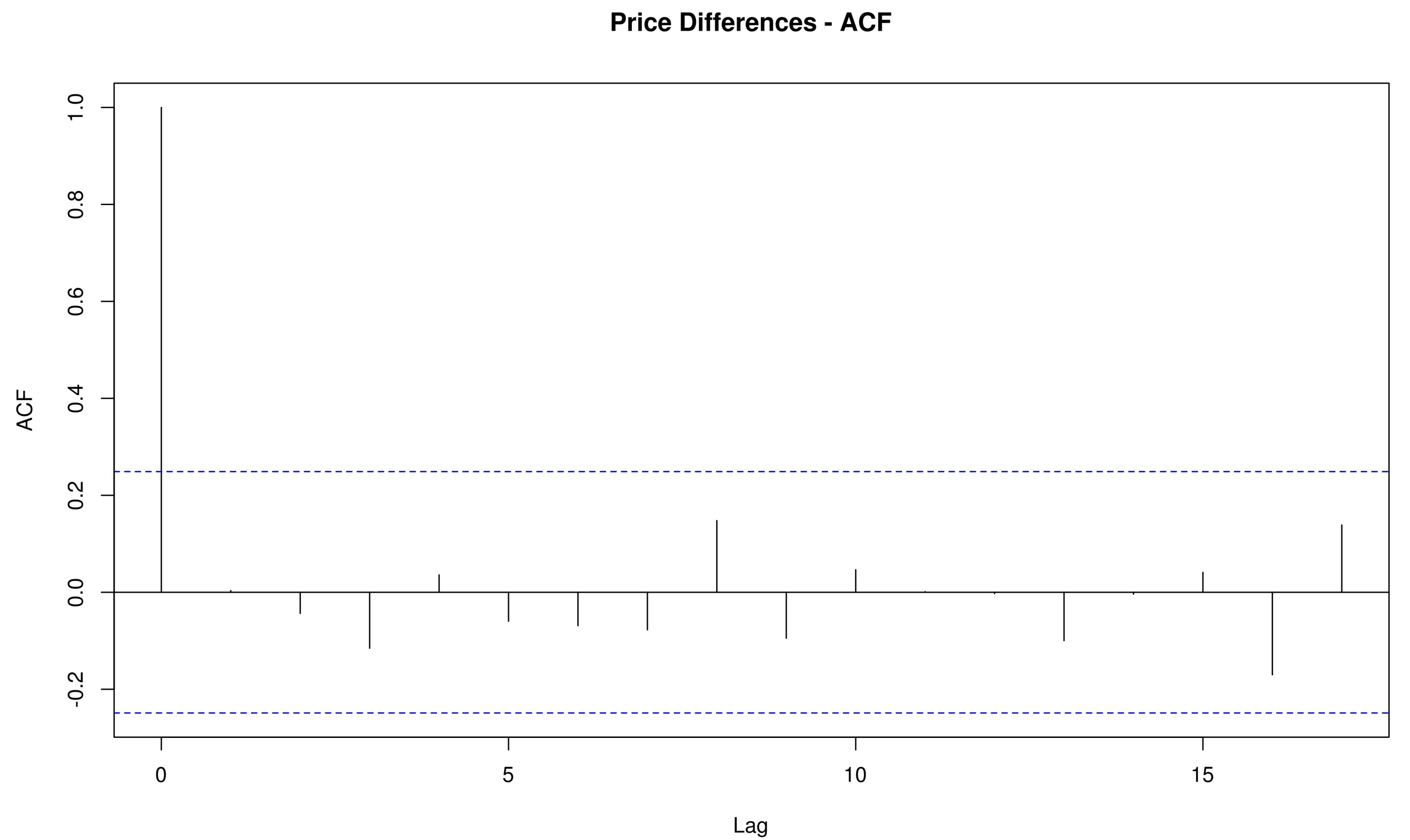
[1] 1.95

Naïve bootstrap example: Stock price, differences, net change

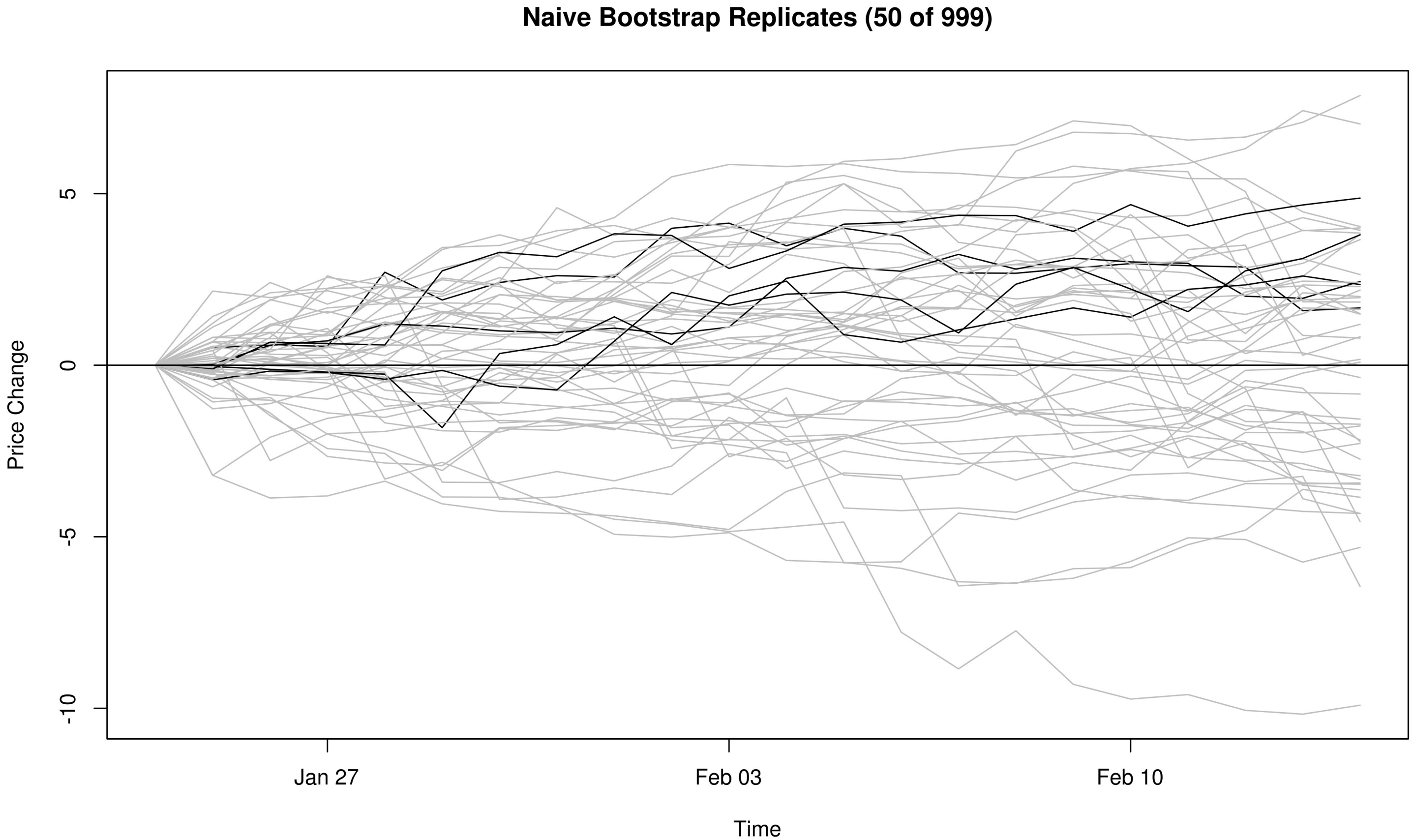




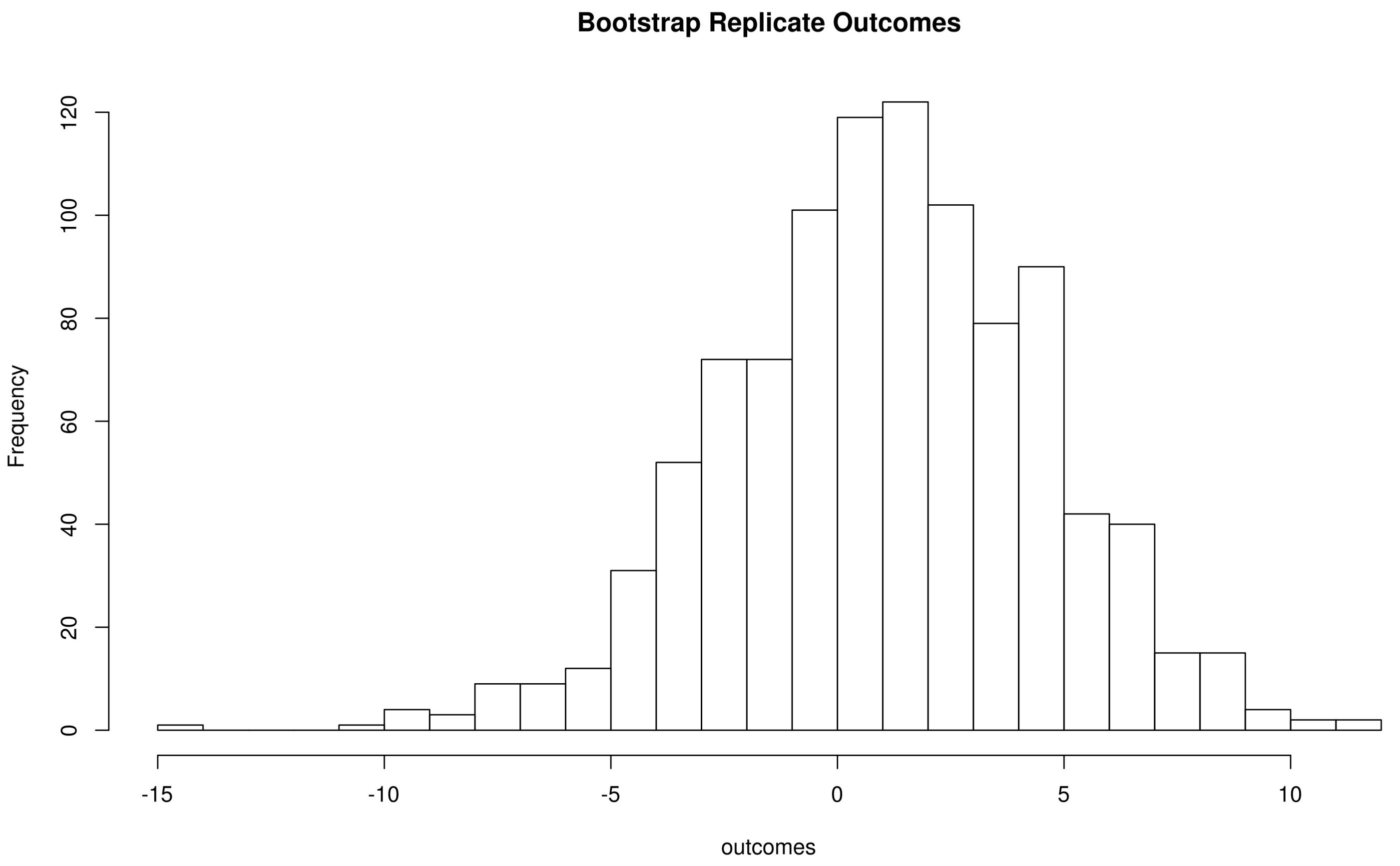
Check: Is it reasonable to assume differences are i.i.d.?



Create replicates by summing resampled differences



Naïve bootstrap example: Statistic of interest is *net change*



Simple implementation in R

```
diffs = diff(price)

HOR = 21

reps = replicate(999,

          sample(diffs, HOR, replace=TRUE),

          simplify=TRUE)

reps = apply(reps, 2, cumsum)

outcomes = reps[HOR,]

print(

  quantile(outcomes, prob=c(0.025, 0.975)) )
```

Mean and quantiles of replicate statistics give estimate and conf. int.

```
> summary(outcomes)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
-14.430  -1.225   1.120   1.057   3.445  11.540
> quantile(outcomes, prob=c(0.025, 0.975))
   2.5%   97.5%
-6.4120  7.6425
```

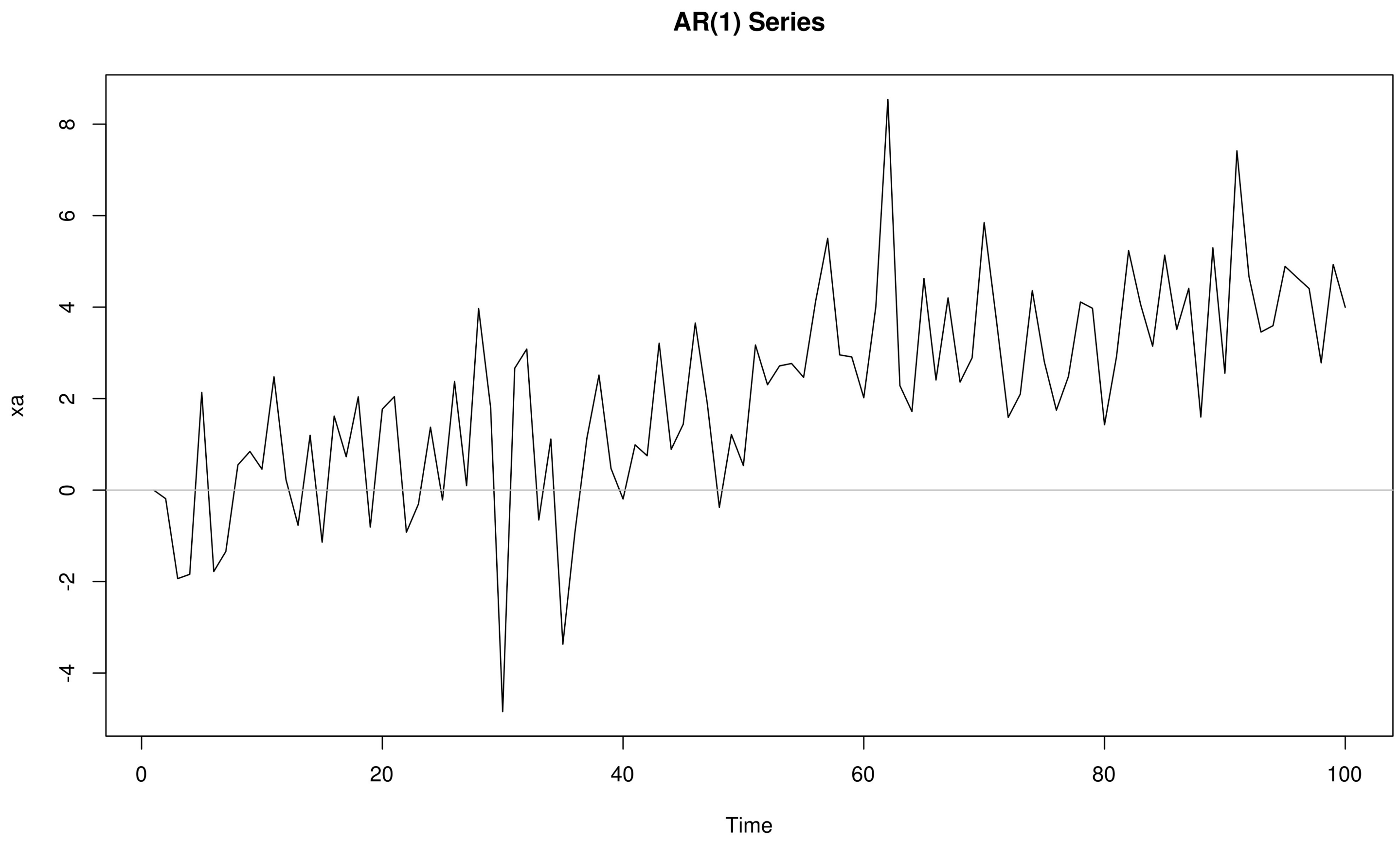
Next problem:
What if the differences are <u>not</u> i.i.d.?

If not, purely random resampling will not capture the structure of the differences.

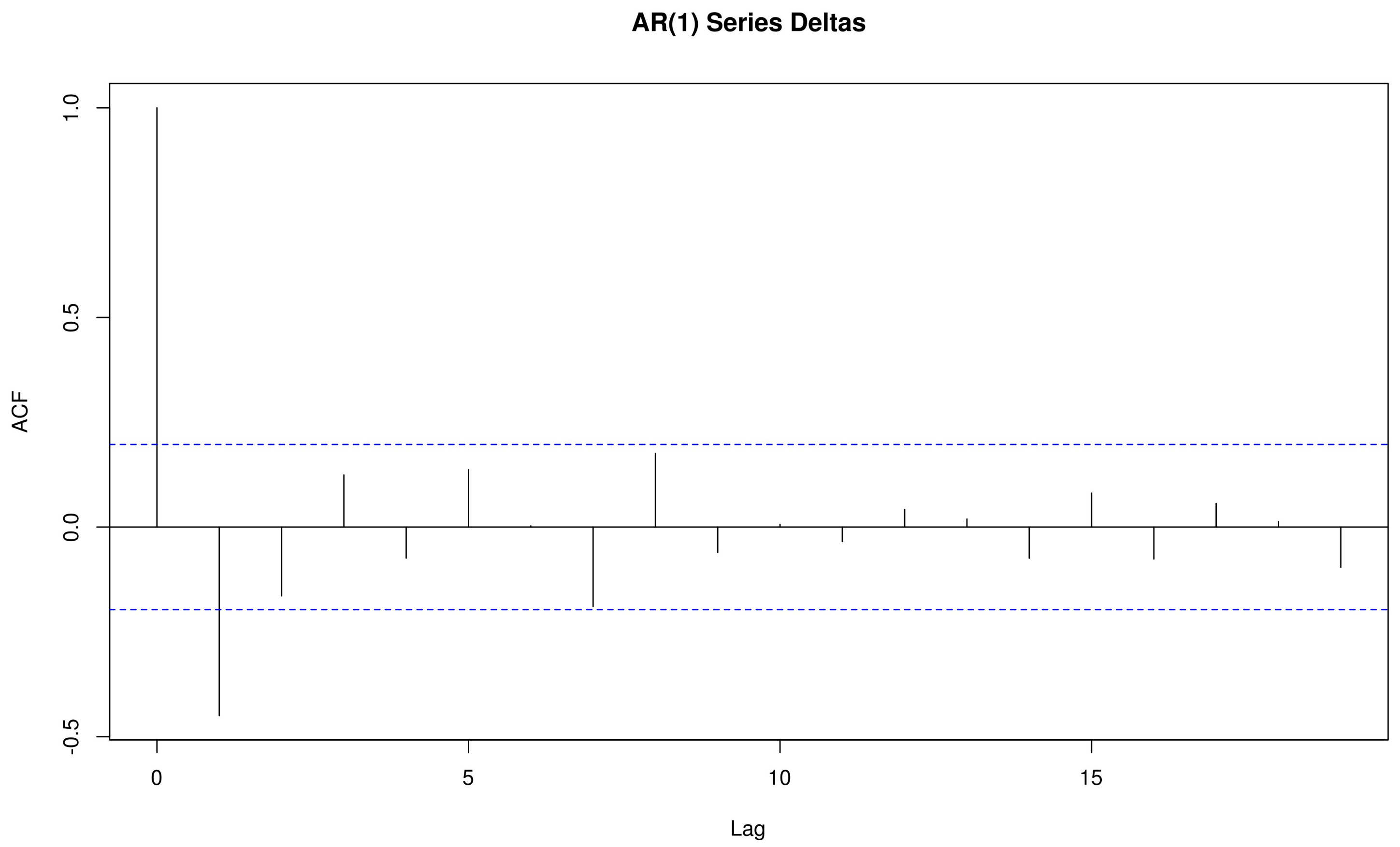
Bootstrap replicates will not resemble our data.

Uh oh.

Example: AR(1) time series, *maximum drawdown* statistic



The ACF of this time series reveals a (simple) dependency.



Moving Block Bootstrap preserves (local) dependency structure.

- Break time series into little blocks.
- Resample the blocks, not individual points – *kind of "random shuffling", with replacement.*
- <u>Within</u> blocks, structure is preserved.
- Works if structure <u>between</u> blocks is (quasi) i.i.d.

The Moving Block procedure resamples blocks, not points.

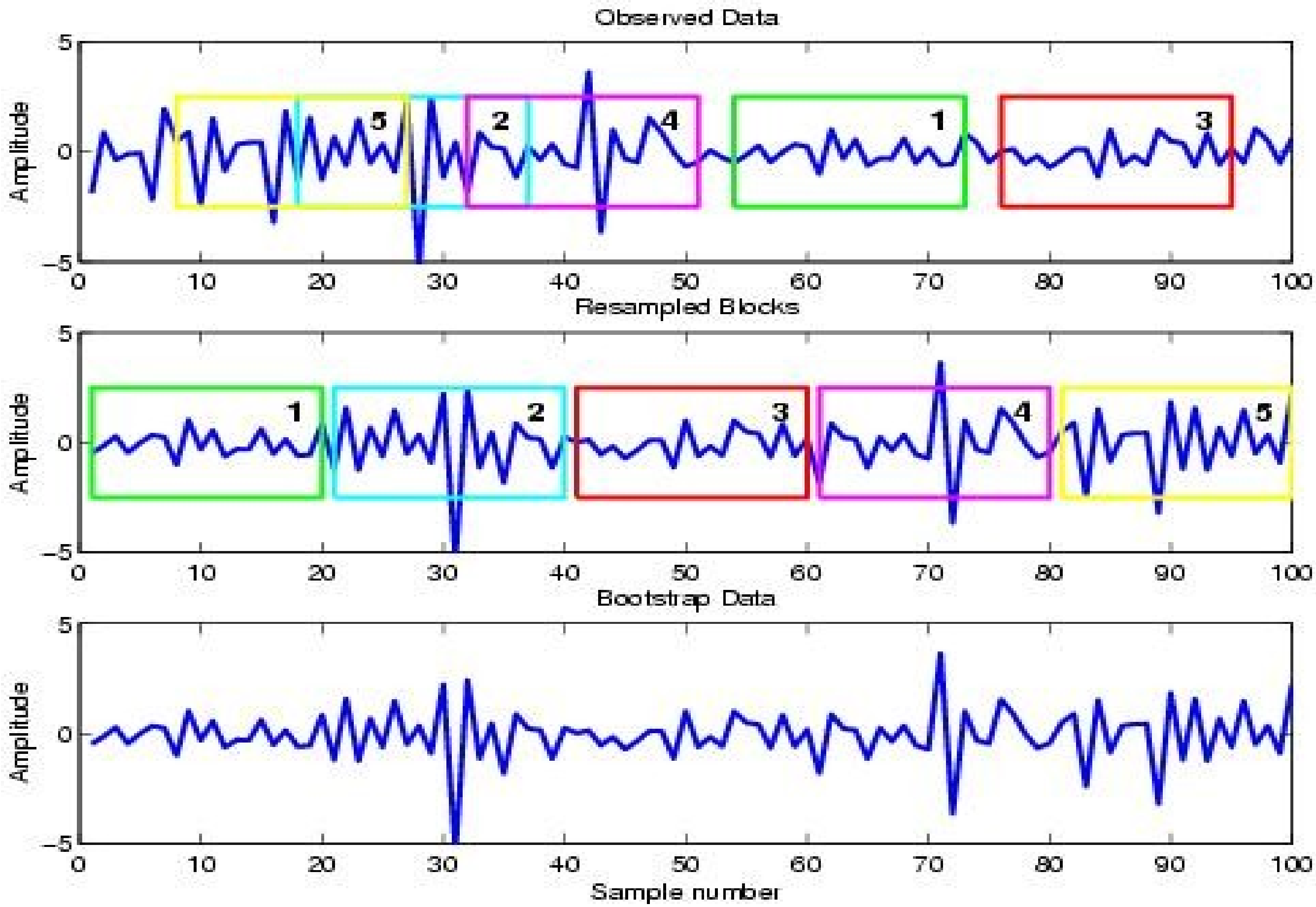


Illustration courtesy of http://www.csp.curtin.edu.au/photos/resample.jpg

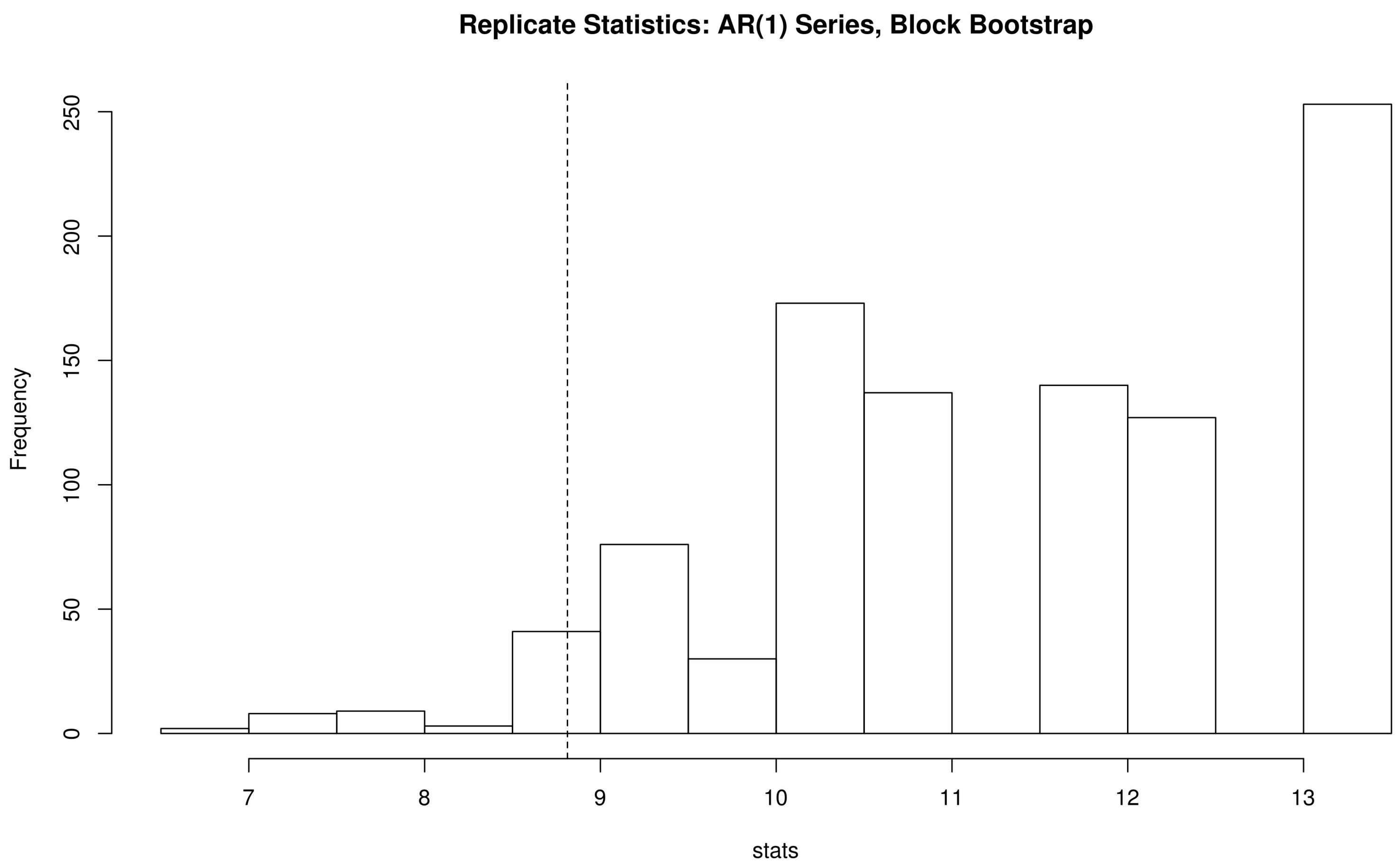
In R, *tsboot* and *boot.ci* together implement a moving block bootstrap.

```
theStatistic = function(x) { . . . }


BLOCK_SIZE = 5        # guess for block size...
mbb = tsboot(ts(xa), theStatistic, R=999,
             l=BLOCK_SIZE, sim="fixed")
replStats = as.vector(mbb$t)
print(summary(replStats))   # for estimate
print(
   boot.ci(mbb, type=c("norm","basic","perc")) )
```

Output from *boot.ci*

```
*** Summary of Replicate Statistics: AR(1) Data, Block Bootstrap
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  6.868  10.350  11.910  11.420  13.390  13.390

*** Confidence Intervals: AR(1) Data, Block Bootstrap
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 999 bootstrap replicates

CALL :
boot.ci(boot.out = bout, type = c("norm", "basic", "perc"))

Intervals :
Level      Normal              Basic              Percentile
95%   ( 3.174,  9.245 )   ( 4.240,  9.084 )   ( 8.541, 13.385 )
Calculations and Intervals on Original Scale
```

Sidebar: Normal approx. does not work for *maximum drawdown*.



What if you have a useful model of your data?

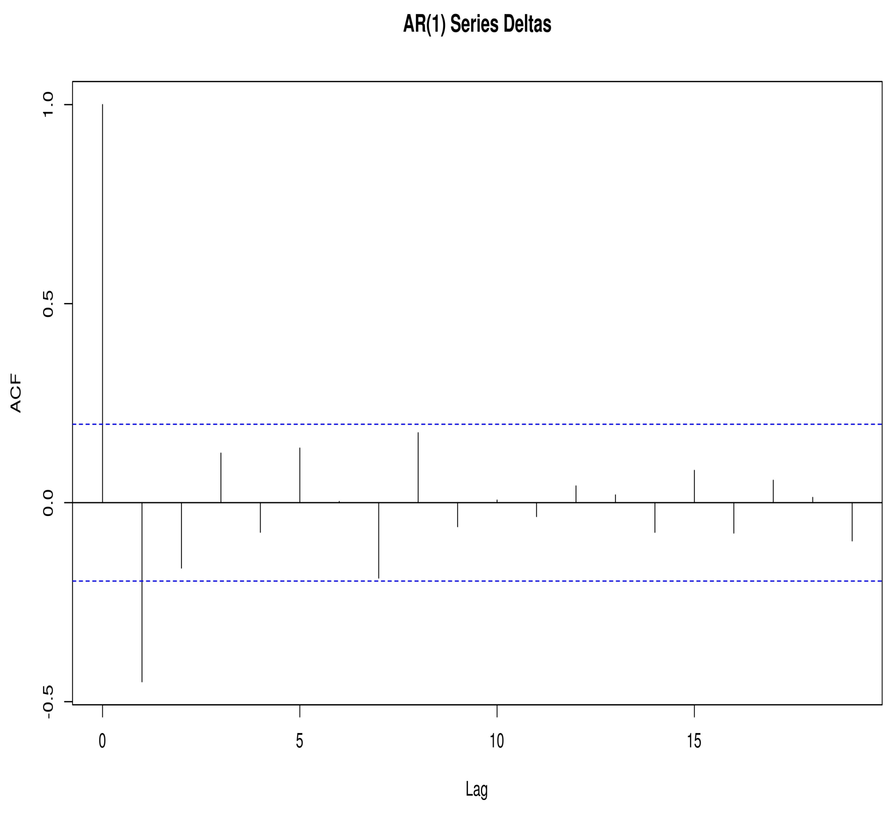
- Example: ARMA, state-space model, or seasonality.
- Model can remove known structure.
- Residuals embody the remaining uncertainty.
- If residuals are i.i.d. time series, we can bootstrap them:

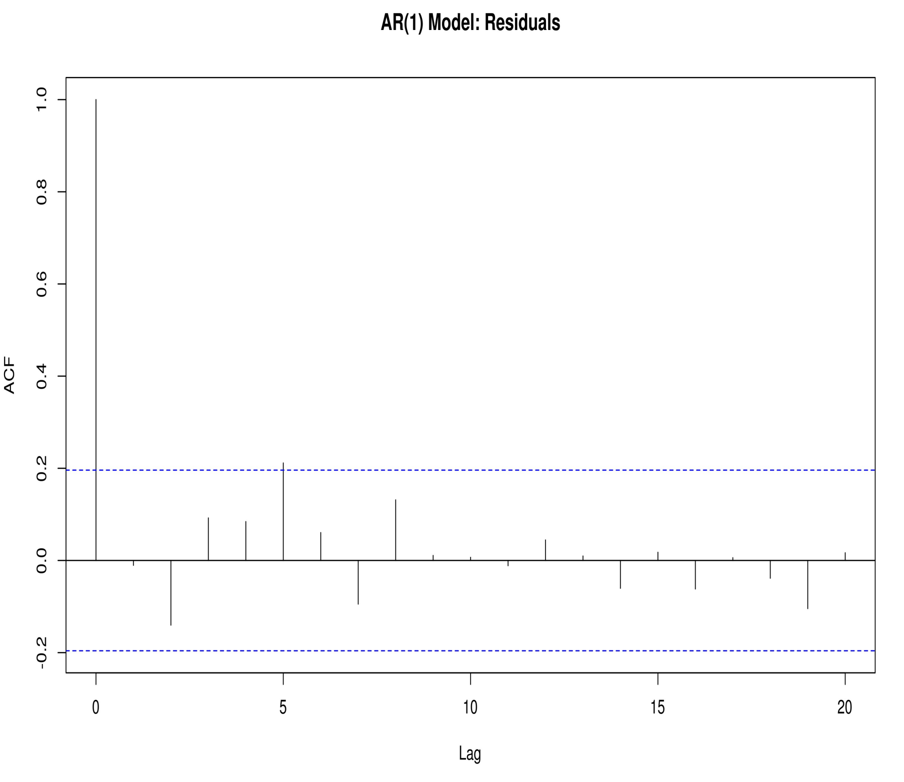
Run the model repeately, each time substituting resampled residuals for originals.

For example, let's fit the AR(1) data to a model (with trend term).

```
*** Fitted AR(1) model:

Call:
arima(x = as.ts(xa), order = c(1, 0, 0), xreg = time,
      include.mean = FALSE)

Coefficients:
          ar1    time
      -0.0329  0.0449
s.e.   0.0995  0.0027

sigma^2 estimated as 2.622:  log likelihood = -190.09,  aic
= 386.18
```

Unlike the original AR(1) data, the residuals show no autocorrelation.





Bootstrap residuals by resampling & inserting them into AR(1) process.

If residuals are

$$\varepsilon_1 \ldots \varepsilon_T$$

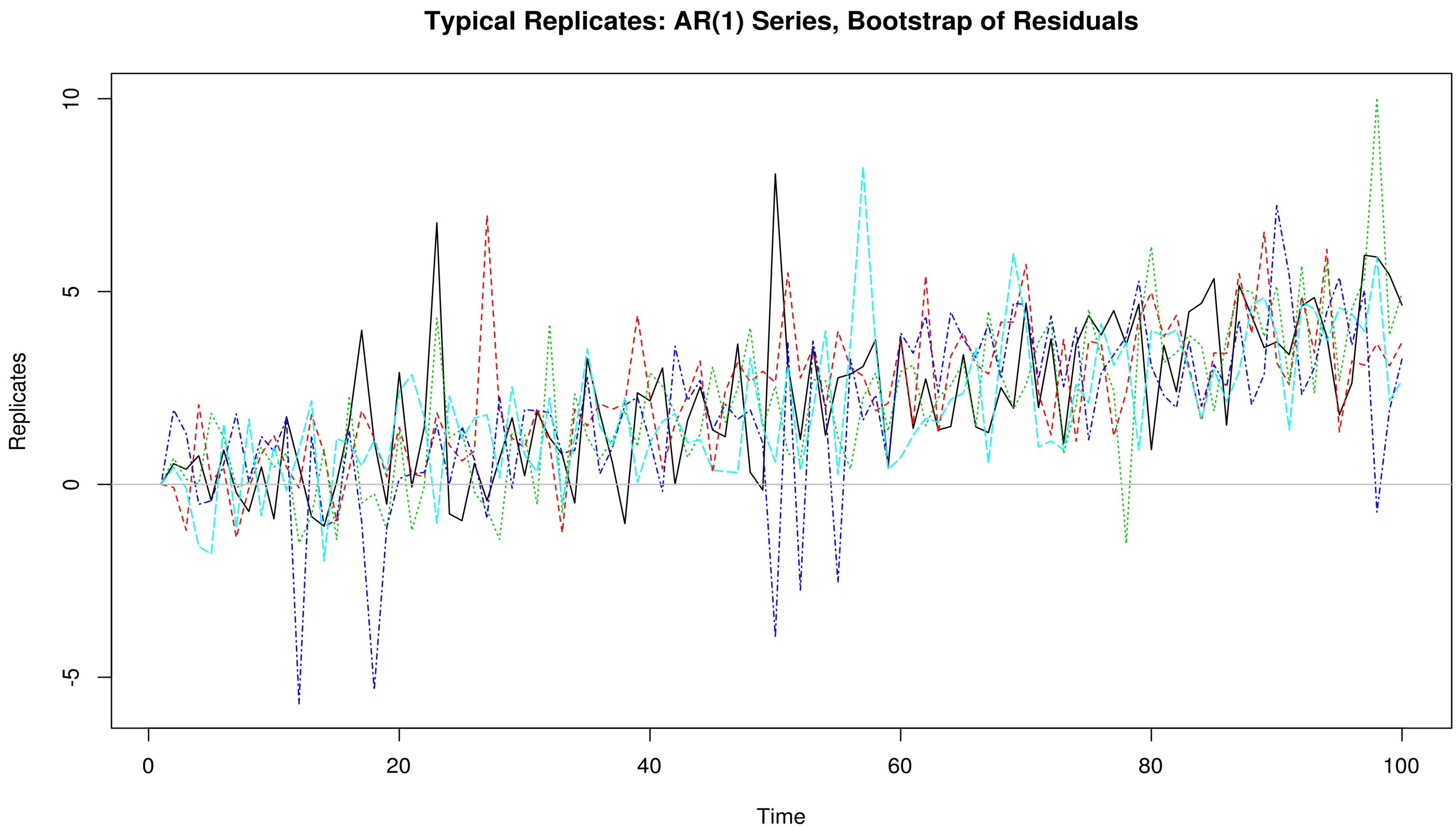
Resample with replacement, giving

$$\varepsilon_1' \ldots \varepsilon_T'$$

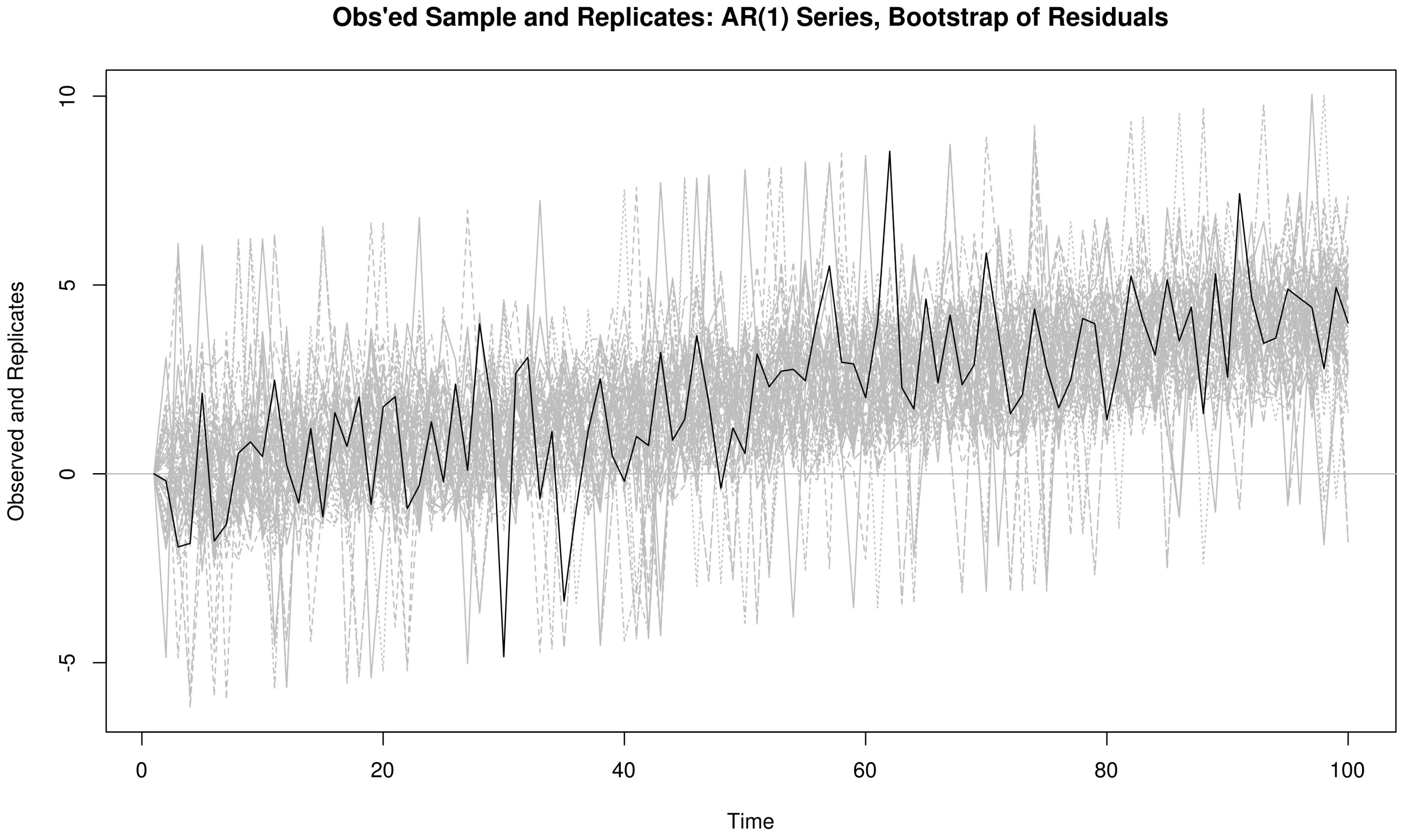
And substitute into the AR(1) process:

$$y_t = \delta + \varphi y_{t-1} + \varepsilon_t'$$

Bootstrap replicates will be plausible variations that conform to the model.



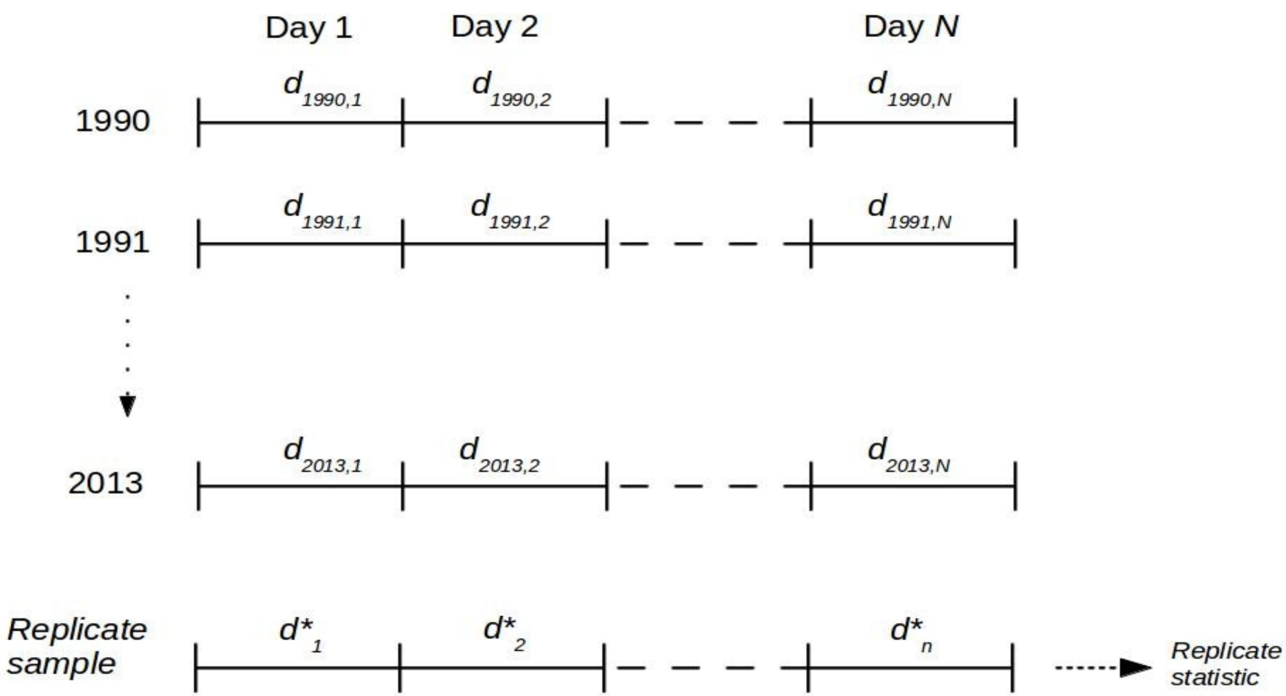
Results of bootstrapping AR(1) residuals



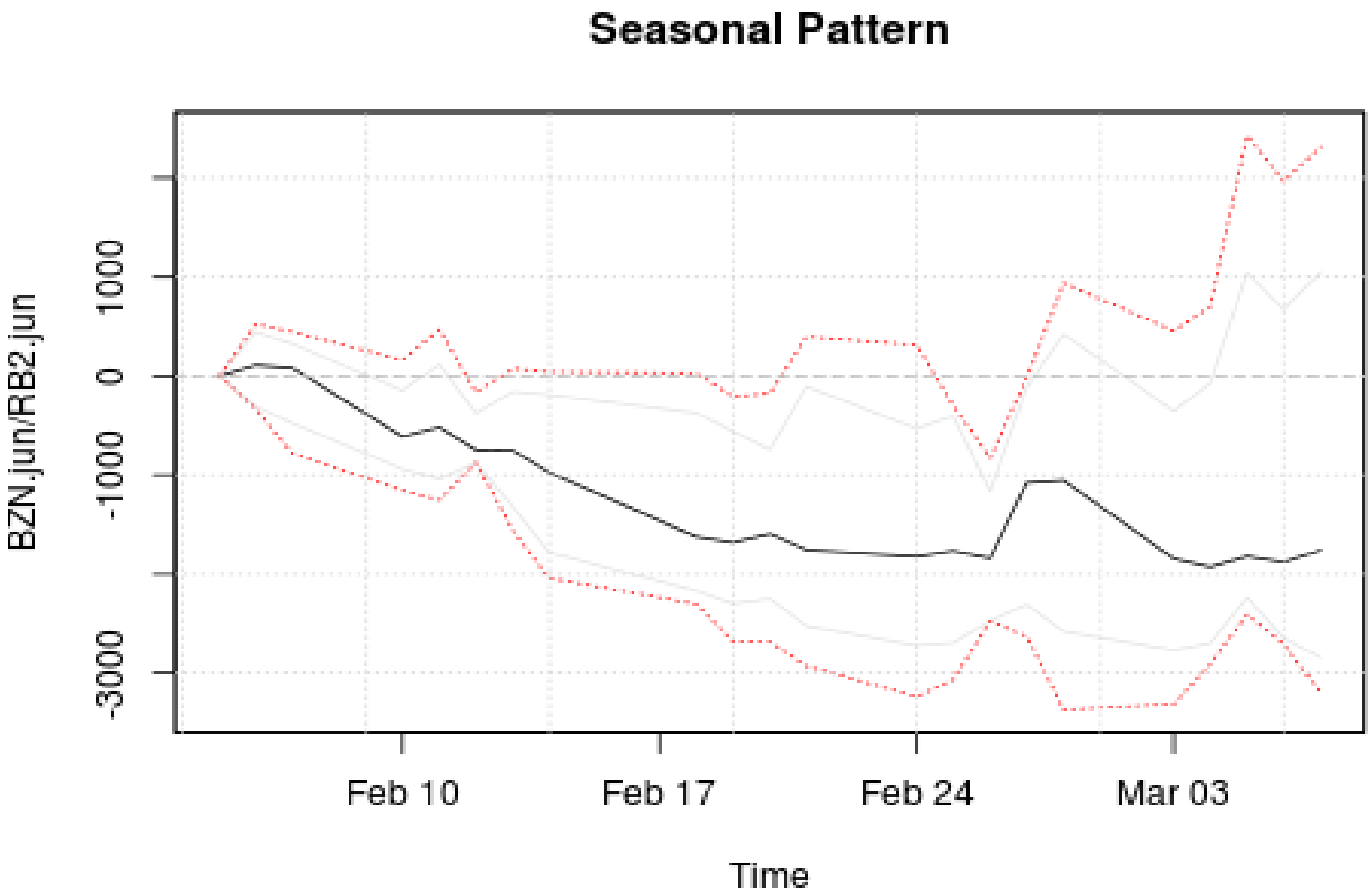
If the model's good, it can tighten the final confidence interval.

```
*** Summary of Replicate Statistics: AR(1) Series, Bootstrap of Residuals
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  3.933   7.784   8.767   8.940  10.410  12.330

*** Confidence Intervals: AR(1) Series , Bootstrap of Residuals
BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
Based on 999 bootstrap replicates

CALL :
boot.ci(boot.out = bout, type = c("norm", "basic", "perc"))

Intervals :
Level      Normal              Basic               Percentile
95%   ( 5.302, 12.067 )   ( 5.816, 12.507 )   ( 5.118, 11.809 )
Calculations and Intervals on Original Scale
```

Seasonality model suggests resampling across seasons.



Seasonal replicates example: median and 95% conf. bands



Advanced techniques can handle other dependency structures.

Procedure	**Structure**
Moving block	Stationary; discrete or categorical data
Local bootstrap – *Similar to Monte Carlo*	Short-range dependence, mild distributional assumtpion.
Markov bootstrap	Staionary, short-range depdence; discrete or categorical data
Sieve bootstrap	AR(n) models

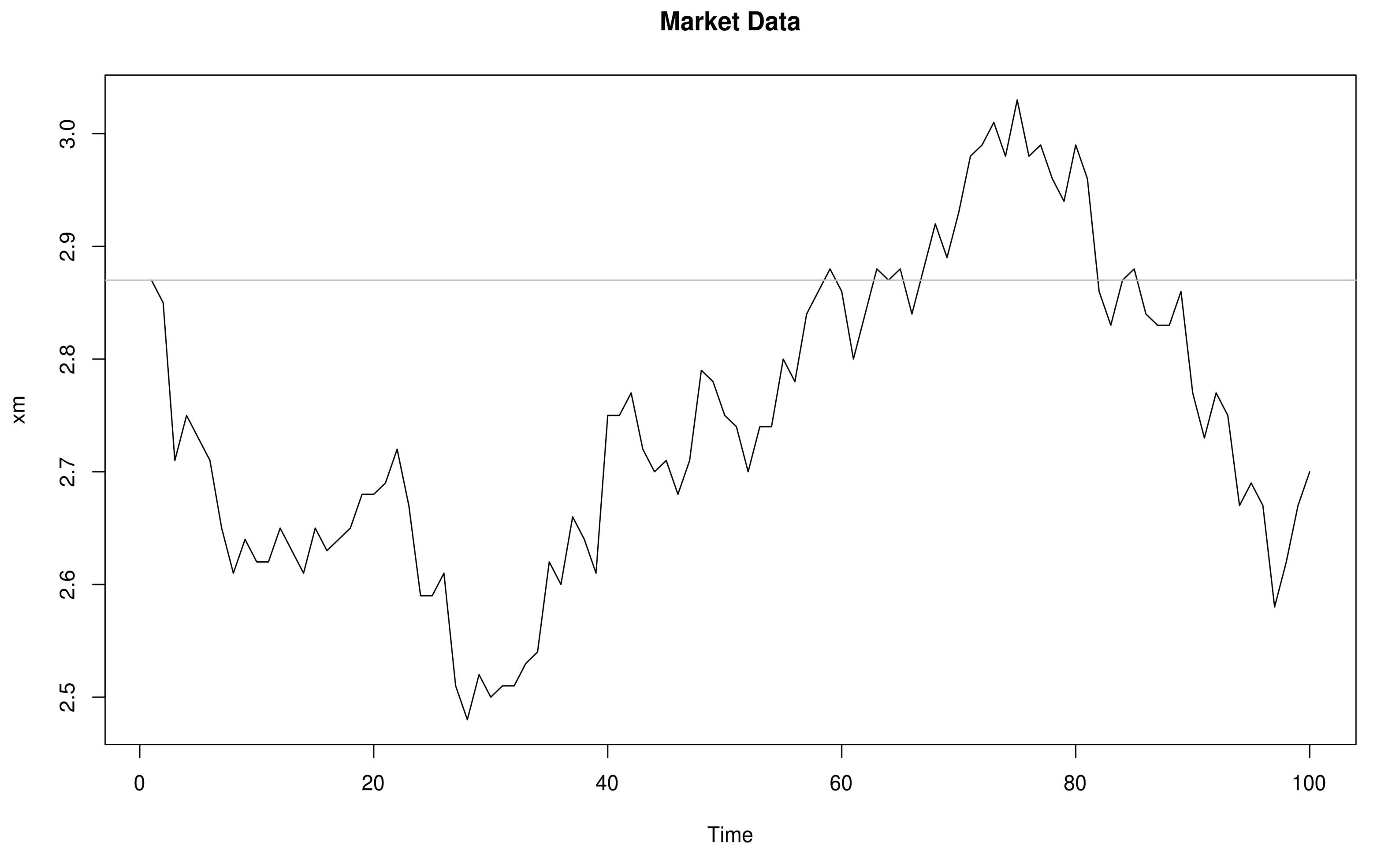
Is there a middle-ground between naïve bootstrap and full model?

- Naïve is, uh, <u>too</u> naïve.
- Model is often unknown.
- Maximum Entropy bootstrap is alternative.
- Parametric bootstrap of differences.
- Maximum entropy distribution of differences – very mild assumption
- Preserves many properties, including shape, seasonality, even some non-stationarity

Vastly oversimplifed outline of maximum entropy bootstrap

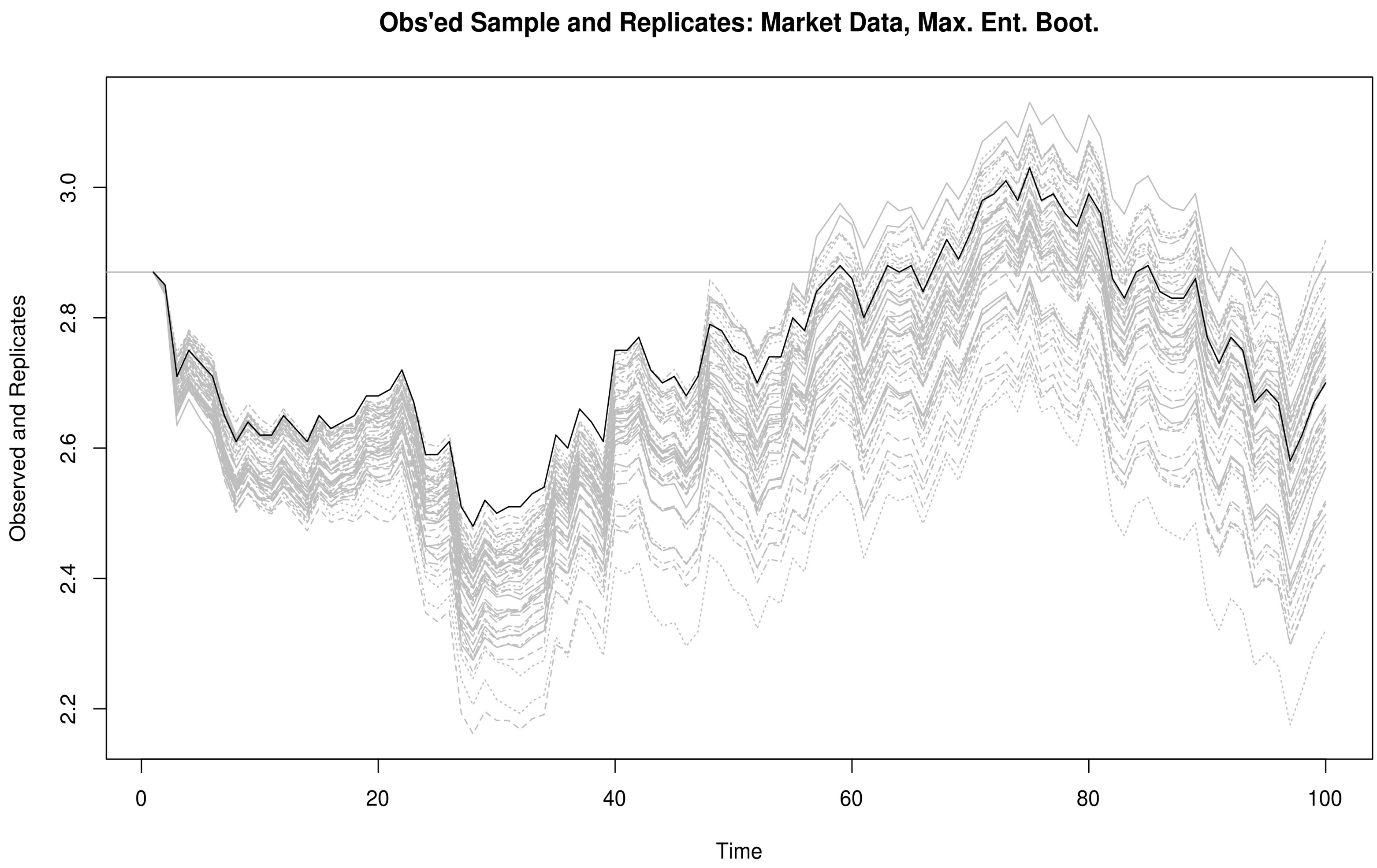
1) Sort the original data.
2) Using sorted data, compute its intermediate points and lower limits for left and right tails.
3) Compute the mean of the maximum entropy density within each interval.
4) Generate uniform random values on [0,1], and compute sample quantiles at those points.
5) Apply to the sample quantiles the correct order to honor the dependence relationships of the observed data.
6) Repeat steps 4 and 5 many times (e.g. 999).

Example: This bond market data seems to have structure. Naive bootstrap works poorly.



The maximum entropy bootstrap preserves the gross structure.

Maximum Entropy Replicates



In R, *meboot* package implements the max. entropy bootstrap.

```
library(meboot)

mebOut = meboot(ts(diff(prices)), reps=999)

mebens = mebOut$ensemble

mebens = rbind(prices[1], mebens)

repls = apply(mebens, 2, cumsum)

# 'repls' contains the bootstrap replicates
```

Bootstrapping Time Series Data: Some Limitations

- Problems with sample: non-representative, too small
- Problems from dependency structure: wrong dependency assumption; regime changes; long-term dependency; overlooked completely
- Parametric bootstrap: wrong model; non-stationary (unstable) process, hence unstable parameters
- Problems with certain statistics: “Edge” statistics may require many, many replicates
- Finally, Monte Carlo may be better alternative

Some References

- *An Introduction to the Bootstrap* by Efron and Tibshirani
- *Bootstrap Methods and Their Applications* by Davison and Hinkley
- "The Moving Blocks Bootstrap Versus Parametric Time Series Models", Vogel and Shallcross, *Water Resources Research* (June 1996)
- "Bootstraps for Time Series", Bühlmann, *Statistical Science* (2002, No. 1)
- "Maximum Entropy Bootstrap for Time Series", Vinod and López-de-Lacalle, *J. of Stat. Soft.* (Jan 2009)

Thank you!

Talk materials available at

`http://bit.ly/csp2014-teetor`