Modeling Proportions and Probabilities: The beta distribution is your friend

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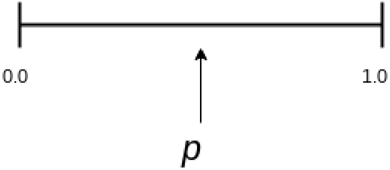
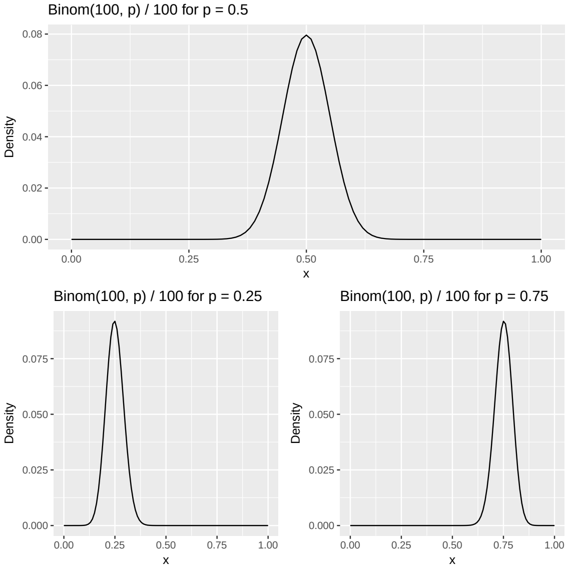
This talk is all about modeling data on the unit interval

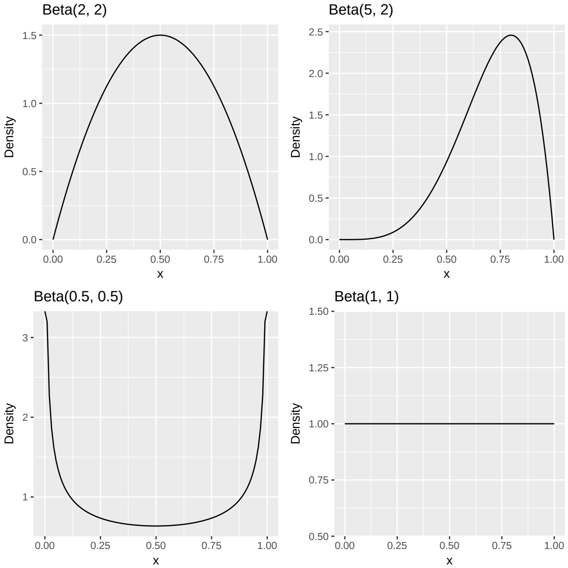


- Proportions (*muggle*: percentages)
- Probabilities
- Continuous ranking (0.0, ..., 1.0)
- Rates, concentrations, etc.

The binomial distribution is a weak model of unit data



The beta distribution adds expressive power



The beta distribution has two parameters

- α: weight on x (for $x \in [0, 1]$)
- β: weight on $1 - x$
- Richer density function than one-parameter binomial model

$$Beta(x|\alpha, \beta) \sim \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

where B is the beta function, providing the normalizing constant

An alternative parameterization uses mean and precision

- *Precision* is reciprocal of variance $= 1/\sigma^2$
- Simple transformation between α and β versus mean (μ) and precision (ϕ)

$$
\begin{aligned}
\mu &= \alpha/(\alpha+\beta) \\
\phi &= \alpha+\beta
\end{aligned}
$$

- Alternative parameterization simplifies regression math

How do we estimate the parameters from sample data?

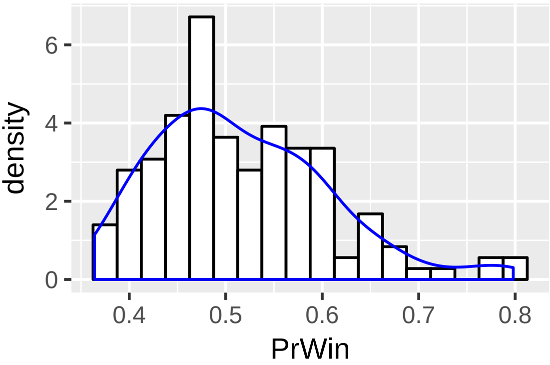
These are the method of moments estimators.

$$\hat{\alpha} = \bar{x}\left(\frac{\bar{x}(1-\bar{x})}{\bar{v}} - 1\right)$$

$$\hat{\beta} = (1-\bar{x})\left(\frac{\bar{x}(1-\bar{x})}{\bar{v}} - 1\right)$$

For data, let's use the Chicago Cubs historical record

```
##   Season PrWin
## 1   1874 0.475
## 2   1875 0.448
## 3   1876 0.788
## 4   1877 0.441
```



Parameter estimation on Cubs data

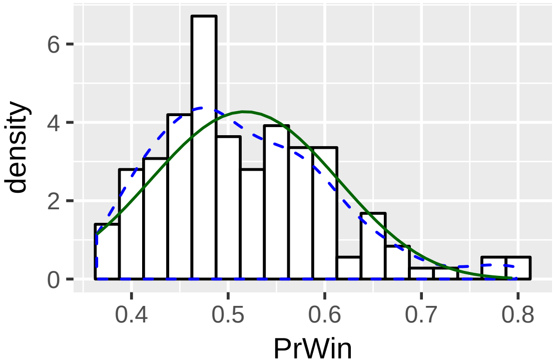
```r
M = mean(cubs$PrWin); V = var(cubs$PrWin)

alpha =  M   *    ((M * (1 - M)) / V - 1)
beta  = (1 - M) * ((M * (1 - M)) / V - 1)

print(alpha);  print(beta)
```

```
## [1] 15.04555
```

```
## [1] 14.10103
```

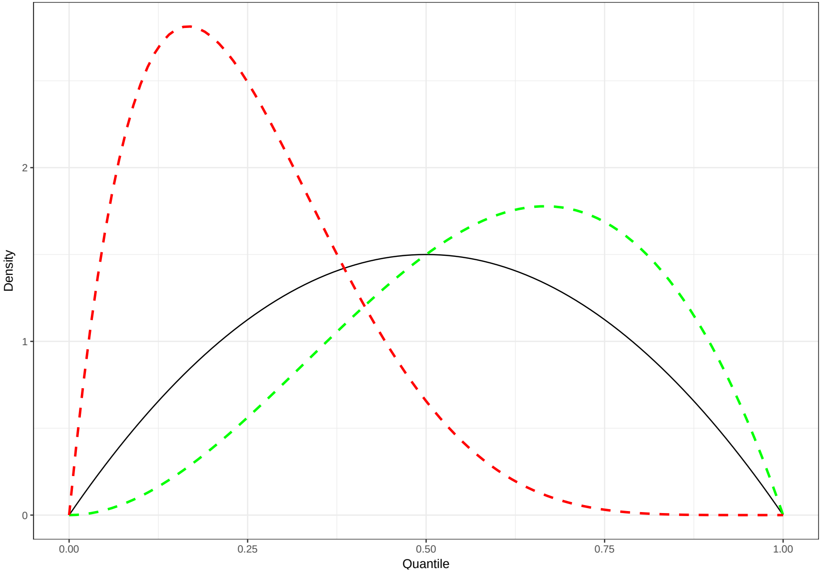
The estimated parameters give a parametric density

The parametric density nicely echos the non-parametric density shown earlier.



Beyond simple fit: Regressors

What if we had a predictor? How would that influence the distribution?



Beta regression models responses with beta distributions

- Response has beta distribution, not normal
- Transform responses from (0,1) to $(-\infty, +\infty)$

$$p_i' = logit(p_i)$$

where $logit(x) = log(\frac{x}{1-x})$

- Then regress on p_i'

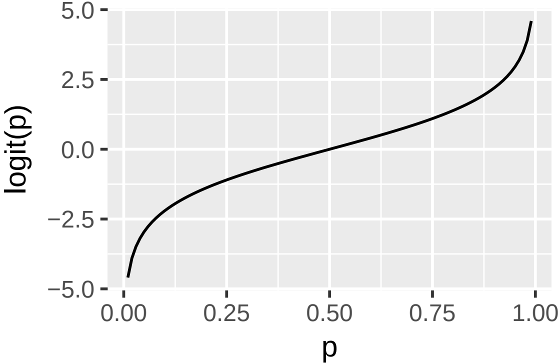
$$p_i' = \beta_0 + \beta_1 x_i + \varepsilon_i$$

- A form of *generalized linear model* (GLM), usually expressed like this

$$logit(p_i) = \beta_0 + \beta_1 x_i + \varepsilon_i$$

The *logit* function mediates between predictors and dependent variable

- Linear predictors range over $(-\infty, +\infty)$
- Dependent variable ranges over bounded (0, 1)
- Logit link function “expands” dependent into $(-\infty, +\infty)$



Beta regression: the Cubs data

- Could this year's success predict next year's success?
- Add column to data: next year's *PrWin*

```
##   Season PrWin PrWinNext
## 1   1874 0.475     0.448
## 2   1875 0.448     0.788
## 3   1876 0.788     0.441
## 4   1877 0.441     0.500
## 5   1878 0.500     0.582
```

Beta regression in R is pretty easy

```r
install.packages("betareg")     # One-time install
```

```r
library(betareg)
```

This example takes the data from the `cubs` data frame.

```r
m = betareg(PrWinNext ~ PrWin, data=cubs)
```

The returned model, `m`, includes β_0 and β_1.

Beta regression: the Cubs data

```r
library(betareg)
m = betareg(PrWinNext ~ PrWin, data=cubs)
print(m)
```

```
##
## Call:
## betareg(formula = PrWinNext ~ PrWin, data = cubs)
##
## Coefficients (mean model with logit link):
## (Intercept)        PrWin
##      -1.059        2.191
##
## Phi coefficients (precision model with identity link):
## (phi)
## 39.76
```

In typical R fashion, summary gives the details (1/2)

```
summary(m)
```

```
##
## Call:
## betareg(formula = PrWinNext ~ PrWin, data = cubs)
##
## Standardized weighted residuals 2:
##     Min      1Q  Median      3Q     Max
## -2.8089 -0.6272 -0.1244  0.6003  4.3545
##
## Coefficients (mean model with logit link):
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  -1.0589     0.1546  -6.847 7.53e-12
## PrWin         2.1912     0.2964   7.393 1.43e-13
```

summary (2/2)

```
## PrWin       ***
##
## Phi coefficients (precision model with identity link):
##       Estimate Std. Error z value Pr(>|z|)
## (phi)   39.758      4.661   8.531   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
##
## Type of estimator: ML (maximum likelihood)
## Log-likelihood: 161.1 on 3 Df
## Pseudo R-squared: 0.2792
## Number of iterations: 15 (BFGS) + 3 (Fisher scoring)
```

Forecast from the model using `predict`

- Create a data frame, `pred`, containing the new predictor values.
- Use `predict` to feed new predictors into model.
- The `type` argument controls what's returned

```r
predict(m, newdata=pred, type="response")
predict(m, newdata=pred, type="variance")
predict(m, newdata=pred, type="quantile",
        at=c(0.025, 0.975))
```

- `predict` handles details of link transformation.

Let's forecast 2017 performance using `predict`

```r
pred = data.frame(PrWin = 0.640)  # Record for 2016
predict(m, newdata=pred)          # Expectation for 2017
```

```
##        1
## 0.585039
```

```r
sqrt(predict(m, newdata=pred, type="variance"))
```

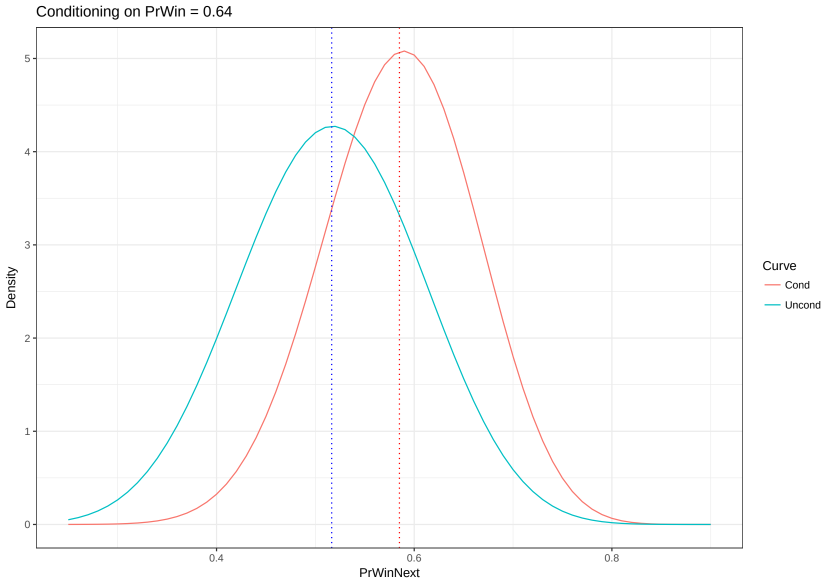
```
##          1
## 0.07717715
```

Let's forecast 2017 performance using `predict` (con't)

```r
predict(m, newdata=pred, type="quantile",
        at=c(0.025, 0.975))
```

```
##        q_0.025  q_0.975
## [1,] 0.4306028 0.731366
```

The regression yields a conditioned beta distribution



The `betareg` package provides other functions, too

```r
methods(class="betareg")
```

```
##  [1] coef             cooks.distance gleverage
##  [4] hatvalues        logLik         model.frame
##  [7] model.matrix     plot           predict
## [10] print            residuals      summary
## [13] terms            update         vcov
## see '?methods' for accessing help and source code
```

Methods for `tidy`, `augment`, and `glance` available from broom package

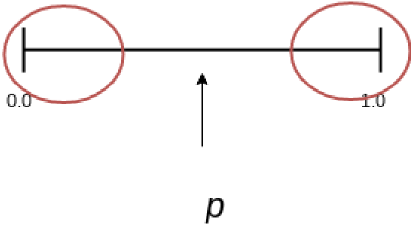
Way cool: We can model precision, too

- Recall that precision is inverse of variance ($1/\sigma^2$)
- Model the precision parameter (ϕ) with a *second* GLM
- Example: suppose z_i predicts variance of p_i

$$logit(\mu_i) = x_i^\top \beta$$
$$log(\phi_i) = z_i^\top \gamma$$

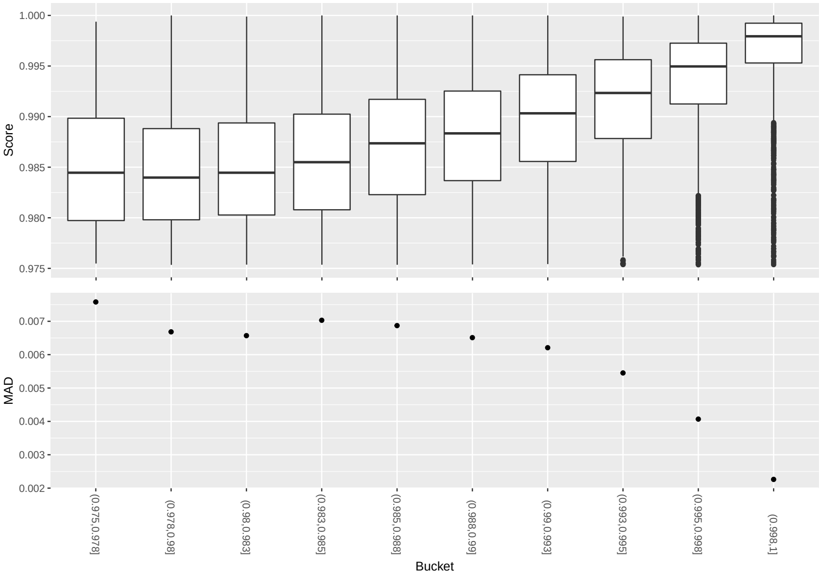
- Incorporates heteroskedasticity into your model
- Small downside: Residuals will be heteroskedastic, so use *standardized weight residuals*

Typical case: Heteroskedasticity at extremes of the unit interval



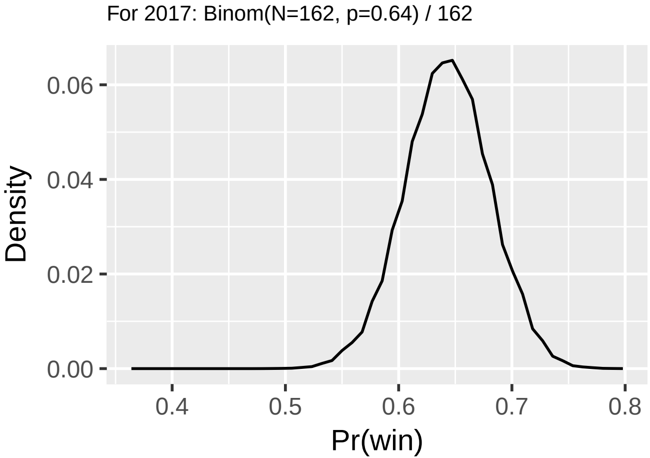
- In my experience, variance can change from midpoint to end-points
- Suggests modeling ϕ_i
- Select z_i best able to model precision, ϕ_i

Heteroskedasticity illustration



So, why doesn't the *binomial* model work?

- Using a simple point estimate of p in $Binom(N, p)$ is naive.
- Does not reflect the uncertainty of the estimate.
- Tails of distr. too small



Improve the model by acknowledging the undercertainty

- Model p as a random variable, imperfect estimate
- Still parameter of binomial distribution, but now stochastic
- The result is called a *Beta-Binomial* model

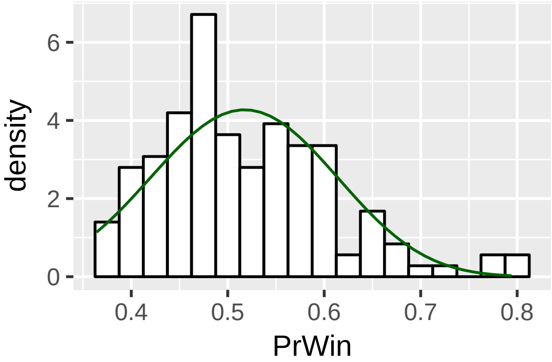
$$PrWin \sim Binom(N, p)$$

where

$$p \sim Beta(\alpha, \beta)$$

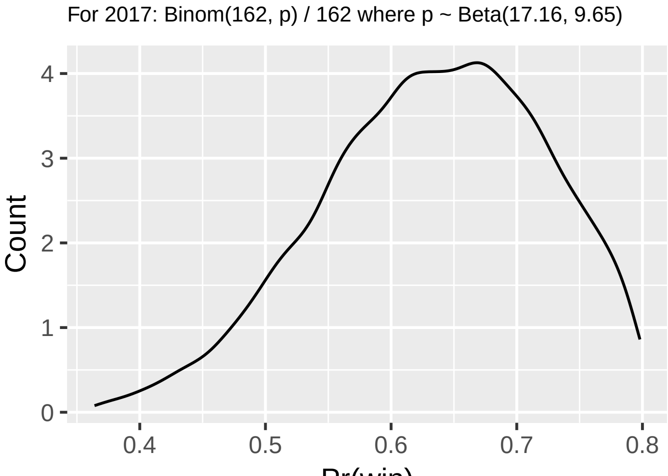
Q: What's the distribution of p?

- Remember, p has a beta distribution.
- Let's estimate mean of p from 2017 record (0.64)
- Estimate variance of p from full history (0.008284)
- From mean and variance, calculate α and β of beta distr.



Beta-binomial distribution for 2017 is more realistic

```
N = 10000
prob = rbeta(N, alpha, beta)
sampWins = rbinom(N, size=SIZE, prob=prob)
sampProbs = sampWins / SIZE
```



The *gamlss* package can fit a beta-binomial model

```
install.packages("gamlss")
```

Beta-binomial model with two predictors, *pred1* and *pred2*:

```
library(gamlss)
m = gamlss(resp ~ pred1 + pred2,
           family = BB, data = yourData)
```

Add a variance predictor, *pred3* (for heteroskedasticity):

```
m = gamlss(resp ~ pred1 + pred2, sigma.formula = ~pred3,
           family = BB, data = yourData)
```

Which is better?

Beta regression:

- Easy. *Hey, there's an R package!*
- Resembles familiar linear regression
- Tools to compare models: log-likelihood, pseudo R^2

Beta-binomial:

- Bayesian, so you get explicit probability distributions, credible intervals
- *gamlss* package does the heavy lifting
- Straight-forward simulation or Monte Carlo

Links to resources

- *betareg* package: https://cran.r-project.org/package=betareg
- *betareg* vignette: https://cran.r-project.org/web/packages/betareg/vignettes/betareg.pdf
- *gamlss* package: https://cran.r-project.org/package=gamlss
- Dave Robinson's excellent beta-binomial tutorial: http://varianceexplained.org/r/beta_binomial_baseball/